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H^2/H^∞ Controller Design for a Two-Dimensional Thin Airfoil Flutter Suppression

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In this paper we study the problem of active feedback controller design for a thin airfoil, whose mathematical model is derived from classical Theodorsen's formulation. A finite dimensional controller stabilizing the original infinite dimensional model is obtained using H^∞ control techniques. We also consider the gust alleviation problem and show that it can be formulated as a disturbance attenuation problem in the mixed H^2/H^∞ control framework. We use existing results on H^∞ and mixed H^2/H^∞ control to illustrate our approach with a numerical example.

I. Introduction

IN this paper we consider the problem of active feedback controller design for a thin airfoil in unsteady aerodynamics. The mathematical model we adopt for this system is obtained from Theodorsen's formulation. Using the H^∞ control theory we derive a finite dimensional feedback controller stabilizing this infinite dimensional model. Then we study the gust alleviation problem and show that it can be seen as a disturbance attenuation problem, which can be solved using the mixed H^2/H^∞ control methods.

In general, mathematical models for airfoils in unsteady aerodynamics are linear time invariant infinite dimensional systems. The basic difficulty in such systems is to compute the aerodynamic loads due to unsteady flow. The simplest models (in the frequency domain) for the unsteady aerodynamics contain Theodorsen's function as the infinite dimensional part. Because of the interaction between the structure and the flow, flutter (dynamic instability) occurs at a certain flow speed. Therefore, it is important to design active feedback controllers stabilizing the airfoil. Gust can be seen as a perturbation in the flow, and its effect on the airfoil can be modeled as an external disturbance. In our feedback controller design, besides closed-loop stability, we would like to reduce the effect of this external disturbance on the system response.

There are several techniques for designing feedback controllers directly from the infinite dimensional airfoil model.[1] In this method the controller itself is infinite dimensional, and hence one has to approximate it to obtain an implementable finite dimensional controller. Another method is to approximate the infinite dimensional part of the system and design a finite dimensional controller from the finite dimensional approximate model.

Most of the work found in the literature on active flutter suppression for Theodorsen's model of a thin airfoil uses the second method. In other words, Theodorsen's function (or the system model generating aeroelastic loads) is approximated first, and a controller is designed (based on several different control techniques) from this finite dimensional model (see Refs. 2–4 and also references therein). However, to our knowledge, the robustness of these controllers to approximations of Theodorsen's function has not been studied in the flutter suppression and gust alleviation context. One has to make sure that the finite dimensional controller designed from the approximate model behaves as desired for the original infinite dimensional model that contains Theodorsen's function. This problem will be studied in the paper, and a procedure to obtain an appropriate controller will be given. The basic idea here is to use H^∞ and mixed H^2/H^∞ control techniques. The H^∞ part of this problem is to guarantee closed-loop stability in the presence of unmodeled dynamics. The H^2 part of it deals with the issue of minimizing the energy of the system response due to gust.

The paper is organized as follows. In the next section we define the mathematical model to be considered for an airfoil in unsteady flow. In Sec. III we present an H^∞ control approach to flutter suppression in the presence of unmodeled aerodynamics. Section IV contains a discussion on how to model the gust and how to put the gust alleviation problem in the framework of the mixed H^2/H^∞ control problem. We present a numerical example in Sec. V. Concluding remarks are made in the last section.

II. Mathematical Model for the Airfoil

We consider the following mathematical model[5–7] for the thin airfoil shown in Fig. 1,

$$M_s\ddot{z}(t) + B_s\dot{z}(t) + K_s z(t) = (1/m_s)\,F(t) + Gu(t) \quad (1)$$

where $z(t) = [h(t), \alpha(t), \beta(t)]^T$, and $u(t)$ represents the control input (torque applied at the flap). The scalar m_s denotes the mass of the structure, and the matrices M_s, B_s, K_s, and G are in the following forms:

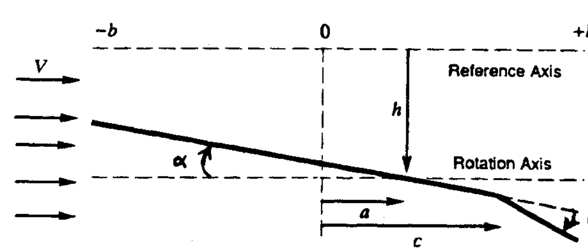

Fig. 1 Thin airfoil.

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$$M_s = \begin{Bmatrix} 1 & bx_\alpha & bx_\beta \\ bx_\alpha & b^2 r_\alpha^2 & b^2[r_\beta^2 + x_\beta(c-a)] \\ bx_\beta & b^2[r_\beta^2 + x_\beta(c-a)] & b^2 r_\beta^2 \end{Bmatrix}$$

$$B_s = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2b^2 r_\beta^2 \zeta_\beta \omega_\beta \end{bmatrix}$$

$$K_s = \begin{bmatrix} \omega_h^2 & 0 & 0 \\ 0 & b^2 r_\alpha^2 \omega_\alpha^2 & 0 \\ 0 & 0 & b^2 r_\beta^2 \omega_\beta^2 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 0 \\ b^2 r_\beta^2 \omega_\beta^2 \end{bmatrix}$$

where x_α, x_β, r_α, r_β, ω_h, ω_α, ω_β, and ζ_β are scalar constants.

In this paper we consider the indicial problem,[1,8] i.e., we assume that $z(t) = 0$ for $t \leq 0$. The aeroelastic loads are represented by $F(t)$, which can be expressed as[4–7,9]

$$F(t) = M_a \ddot{z}(t) + B_a \dot{z}(t) + K_a z(t) + F_c(t) \tag{2}$$

where $F_c(t)$ is the "circulatory" part of $F(t)$. The matrices M_a, B_a, and K_a can be computed in terms of the problem data

$$M_a = -\rho b^2 \begin{Bmatrix} \pi & -\pi ba & -T_1 b \\ -a\pi b & \pi b^2 (1/8 + a^2) & -[T_7 + (c-a)T_1]b^2 \\ -T_1 b & -[T_7 + (c-a)T_1]b^2 & -T_3 b^2/\pi \end{Bmatrix}$$

$$B_a = -\rho b^2 V \begin{Bmatrix} 0 & \pi & -T_4 \\ 0 & \pi(0.5-a)b & [T_1 - T_8 - (c-a)T_4 + 0.5T_{11}]b \\ 0 & [T_4(a-0.5) - T_1 - 2T_9]b & -T_4 T_{11} b/2\pi \end{Bmatrix}$$

$$K_a = -\rho b^2 V^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & T_4 + T_{10} \\ 0 & 0 & (T_5 - T_4 T_{10})/\pi \end{bmatrix}$$

where the various T_i are Theodorsen's constants.[6,7]

According to Theodorsen's formulation, $F_c(t)$ can be expressed in the frequency domain as (see Ref. 7, pp. 395, 396 or Ref. 4, pp. 26–28)

$$\hat{F}_c(s) = T(s)(B_{c1} + sB_{c2})\hat{z}(s) \tag{3}$$

where s is the Laplace transform variable, represents the Laplace transformed version of a time signal, $\hat{T}(j\omega)$ is Theodorsen's function, and B_{c1} and B_{c2} are constant matrices given by

$$B_{c1} = b_1 c_1, \qquad B_{c2} = b_1 c_2$$

where

$$b_1 = \rho V b \begin{bmatrix} -2\pi \\ 2\pi b(a + 0.5) \\ -T_{12} b \end{bmatrix}$$

$$c_1 = V[0 \quad 1 \quad T_{10}/\pi], \qquad c_2 = [1 \quad b(0.5-a) \quad bT_{11}/2\pi]$$

Let

$$y(t) := c_1 z(t) + c_2 \dot{z}(t)$$

be the measured output of the system. Then, taking the Laplace transforms of Eqs. (1) and (2) and then using Eq. (3), we obtain a transfer function from u to y, denoted by $P(s)$:

$$\frac{\hat{y}(s)}{\hat{u}(s)} = P(s) = \frac{C_0(sI - A)^{-1}B_0}{1 - C_0(sI-A)^{-1}B_1 T(s)} \tag{4}$$

where $T(s)$ is Theodorsen's function, and

$$A = \begin{bmatrix} 0_{3\times 3} & I_{3\times 3} \\ \left(M_s - \dfrac{M_a}{m_s}\right)^{-1}\left(\dfrac{K_a}{m_s} - K_s\right) & \left(M_s - \dfrac{M_a}{m_s}\right)^{-1}\left(\dfrac{B_a}{m_s} - B_s\right) \end{bmatrix}$$

$$C_0 = [c_1 \quad c_2], \qquad B_1 = \begin{bmatrix} 0_{3\times 1} \\ \left(M_s - \dfrac{M_a}{m_s}\right)^{-1} b_1 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} 0_{3\times 1} \\ \left(M_s - \dfrac{M_a}{m_s}\right)^{-1} G \end{bmatrix}$$

Note that the plant can be seen as a feedback system whose feedback path consists of the aerodynamics represented by Theodorsen's function, as shown in Fig. 2.

In practice, $T(s)$, which represents an infinite dimensional system, is approximated by a low-order rational function, say $T_a(s)$. This leads to a finite dimensional approximate model for the plant to be controlled

$$P_f(s) = \frac{C_0(sI - A)^{-1}B_0}{1 - C_0(sI - A)^{-1}B_1 T_a(s)}$$

Approximation of T by a second-order rational function T_a has been discussed in Ref. 10, and the L^∞ error bounds [i.e., $\sup_\omega |T(j\omega) - T_a(j\omega)|$] in several approximation methods have been compared. In the next section we design a feedback controller for $P_f(s)$, which is guaranteed to stabilize both $P_f(s)$ and $P(s)$. The main tools used are the H^∞ control techniques.

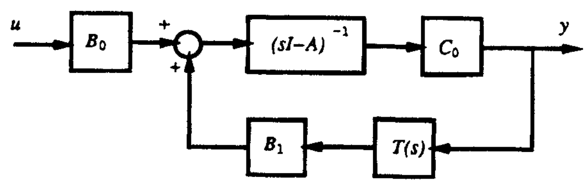

Fig. 2 Structure of the plant.

III. Stabilization by a Finite Dimensional Controller

Let us consider the thin airfoil model obtained earlier. When flutter occurs, the plant $P(s)$ is unstable, and we would like to design a feedback controller $K_f(s)$ stabilizing the closed-loop system formed by $u = -K_f y$. In our design we will consider a stable finite dimensional T_a (see Ref. 10 for the exact numerical values of the coefficients of T_a). This will give us an approximate plant model P_f. A robustly stabilizing finite dimensional controller $K_f(s)$ will be obtained from P_f, and it will be shown that, under a certain condition, this controller stabilizes the original infinite dimensional airfoil model, with a certain robustness level.

Consider the finite dimensional approximate plant

$$P_f(s) = \frac{C_0(sI - A)^{-1}B_0}{1 - C_0(sI - A)^{-1}B_1 T_a(s)}$$

We can find rational transfer functions $N_1, N_2, M \in H^\infty$, such that

$$C_0(sI - A)^{-1}B_0 = \frac{N_0(s)}{M(s)} \qquad C_0(sI - A)^{-1}B_1 = \frac{N_1(s)}{M(s)}$$

Therefore, we can express P and P_f in the form

$$P_f(s) = \frac{N_0(s)}{M(s) - N_1(s)T_a(s)} \qquad P(s) = \frac{N_0(s)}{M(s) - N_1(s)T(s)}$$

Thus, P and P_f differ in their denominator, in the sense that

$$P(s) = \frac{N_p(s)}{M_p(s)} \qquad P_f(s) = \frac{N_p(s)}{M_f(s)}$$

where $N_p(s) = N_0(s)$, $M_p(s) = M(s) - N_1(s)T(s)$, $M_f(s) = M_p(s) + \Delta_M(s)$, and

$$\Delta_M(s) = N_1(s)[T(s) - T_a(s)]$$

Let ϵ_a be an upper bound of the L^∞ approximation error for Theodorsen's function, i.e.,

$$\|T - T_a\|_\infty < \epsilon_a$$

It was shown[10] that a second-order rational function T_a, [see Sec. V for the precise definition of $T_a(s)$] gives an error bound $\epsilon_a = 0.012$.

Lemma 1: A controller K_f stabilizing P_f and achieving an H^∞ performance level

$$\gamma(K_f) = \|N_1 M_f^{-1}(1 + P_f K_f)^{-1}\|_\infty \tag{5}$$

stabilizes the infinite dimensional plant P if

$$\gamma(K_f) \le \frac{1}{\epsilon_a} \tag{6}$$

Proof: A controller K_f stabilizes the closed-loop system with the plant P if and only if[11-13] $S = (1 + PK_f)^{-1}$, PS, $K_f S \in H^\infty$. Since K_f stabilizes P_f, it has to be in the form[11-13] $K_f = (X + M_f Q)/(Y - N_p Q)$ for some $X, Y, Q \in H^\infty$. Then the sensitivity function of the feedback system with controller K_f and the actual plant P is of the form

$$S = \frac{1}{1 + PK_f}$$
$$= \frac{1}{1 + [N_p/(M_f + \Delta_M)][(X + M_f Q)/(Y - N_p Q)]}$$
$$= \frac{(M_f + \Delta_M)(Y - N_p Q)}{(M_f + \Delta_M)(Y - N_p Q) + N_p X + N_p M_f Q}$$
$$= \frac{(M_f + \Delta_M)(Y - N_p Q)}{1 + \Delta_M(Y - N_p Q)}$$

From this expression we get

$$K_f S = \frac{(M_f + \Delta_M)(X + M_f Q)}{1 + \Delta_M(Y - N_p Q)}$$

$$PS = \frac{N_p(Y - N_p Q)}{1 + \Delta_M(Y - N_p Q)}$$

Hence, S, $K_f S$, $PS \in H^\infty$ if[11,13]

$$\|\Delta_M(Y - N_p Q)\|_\infty < 1 \tag{7}$$

Since $|\Delta_M(j\omega)| < \epsilon_a |N_1(j\omega)|$, the inequality (7) holds if

$$\|N_1(Y - N_p Q)\|_\infty \le \frac{1}{\epsilon_a}$$

On the other hand, it is easy to see that

$$N_1(Y - N_p Q) = N_1 M_f^{-1}(1 + P_f K_f)^{-1} \qquad \square$$

The controller K_f^{opt}, which minimizes $\gamma(K_f)$ over all controllers stabilizing P_f, has the best chance of satisfying Eq. (6). That is, if we define

$$\gamma_{\text{opt}} := \inf_{K_f \text{stabilizing } P_f} \gamma(K_f) =: \gamma(K_f^{\text{opt}})$$

then the controller K_f^{opt} stabilizes P_f, and all plants of the form P, whose approximation error ($\epsilon_a = \|T - T_a\|_\infty$) satisfies

$$\epsilon_a \le \gamma_{\text{opt}}^{-1}$$

IV. Gust Alleviation Problem

In this section we would like to outline a mixed H^2/H^∞ control approach to the gust alleviation problem. We can model the gust as a disturbance in the flow. Therefore, gust enters the mathematical model in the computation of aerodynamic loads. We can think that[1] $F(t)$ is perturbed by a term $n_g(t)$ that is the output of a filtered white noise, i.e.,

$$n_g(t) = \int_0^t W_g(t - \tau)w(\tau)\,d\tau$$

where w is white Gaussian with unit spectral density, and $\hat{W}_g(s)$ is a 3×1 filter shaping the spectral density of the gust. The term $n_g(t)$ modifies the equation (1) governing the airfoil motion in such a way that we now have

$$\hat{y}(s) = P(s)\hat{u}(s) + F_g(s)\hat{w}(s)$$

where

$$F_g(s) = \frac{C_0(sI - A)^{-1}A_1 \hat{W}_g(s)}{1 - C_0(sI - A)^{-1}B_1 T(s)}, \qquad \text{with}$$

$$A_1 = \frac{1}{m_s}\begin{bmatrix} 0_{3\times 3} \\ \left(M_s - \dfrac{M_a}{m_s}\right)^{-1} \end{bmatrix}$$

The term $F_g(s)$ can be seen as a filter generating the output disturbance in the system. Therefore, the closed-loop system can be represented as shown in Fig. 3.

We would like to "minimize" the effect of the gust on the system output; i.e., the output energy is to be minimized when the system is excited by the gust. The feedback controller generates the command signal: $\hat{u}(s) = -K_f(s)\hat{y}(s)$. Therefore, we want to minimize the energy of z_{reg}: $= y$ over all controllers K_f stabilizing the previous closed-loop system. That is, we want to minimize

$$\gamma_2(K_f) := \|(1 + PK_f)^{-1}F_g\|_2 = \|z_{\rm reg}\|_2 \tag{8}$$

while keeping the closed-loop system stable. It can be shown that the previous feedback system is equivalent to the system shown in Fig. 4, where $\Delta_T(s) = T(s) - T_a(s)$.

Now consider the closed-loop system shown in Fig. 5, where the outer loop containing Δ_T in Fig. 4 has been ignored. Furthermore, two external inputs are added (taking measurement noise and the approximation error into account): n is a white noise uncorrelated with w, and v is deterministic and its energy is bounded. The filter $W_n(s)$ is a weighting function for n. In this system the output to be regulated is denoted by z_1, and the command signal is represented by u_1.

After some algebraic manipulations, it can be shown that if $v = n = 0$ then

$$z_{\rm reg} = \frac{1}{1 - \dfrac{C_0(sI - A)^{-1}B_1\Delta_T(s)}{1 + C_0(sI - A)^{-1}B_0K_f(s) - C_0(sI - A)^{-1}B_1T_a(s)}} z_1$$

Therefore, in the case of $v = n = 0$ we have

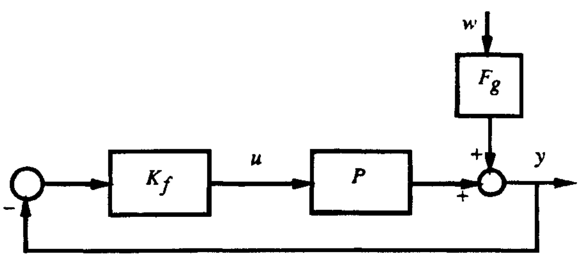

Fig. 3 Feedback system with output disturbance.

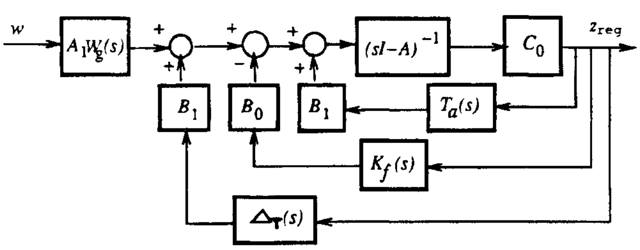

Fig. 4 Equivalent feedback system.

$$\|z_{\rm reg}\|_2 \leq \frac{1}{1 - \epsilon_a \left\| \dfrac{\{[C_0(sI - A)^{-1}B_1]}{[1 + C_0(sI - A)^{-1}B_0K_f(s) - C_0(sI - A)^{-1}B_1T_a(s)]\}} \right\|_\infty} \|z_1\|_2 \tag{9}$$

Note that

$$\left\| \frac{C_0(sI - A)^{-1}B_1}{1 + C_0(sI - A)^{-1}B_0K_f(s) - C_0(sI - A)^{-1}B_1T_a(s)} \right\|_\infty = \|N_1M_f^{-1}(1 + P_fK_f)^{-1}\|_\infty = \gamma(K_f)$$

Hence, we conclude that if

$$\epsilon_a\gamma(K_f) \ll 1$$

then

$$\|z_{\rm reg}\|_2 \approx \|z_1\|_2$$

Let us now consider the case $v \neq 0 \neq n$. It is easy to see that

$$\epsilon_a\gamma(K_f) = \|S_{z_1 \leftarrow v}\|_\infty$$

where $S_{z_1 \leftarrow v}$ denotes the transfer function from v to z_1. Now define

$$\gamma_2' := \|S_{\left[\begin{smallmatrix} z_1 \\ ru_1 \end{smallmatrix}\right] \leftarrow \left[\begin{smallmatrix} w \\ n \end{smallmatrix}\right]}\|_2$$

and

$$\gamma_\infty' := \|S_{\left[\begin{smallmatrix} z_1 \\ ru_1 \end{smallmatrix}\right] \leftarrow v}\|_\infty$$

for some $0 < r \ll 1$.

Lemma 2: A finite dimensional controller K_f stabilizing the finite dimensional feedback system shown in Fig. 5 stabilizes the original infinite dimensional plant P, if $\gamma_\infty' < 1$. Furthermore, this controller guarantees that

$$\|z_{\rm reg}\|_2 \leq \frac{\gamma_2'}{1 - \gamma_\infty'} =: \gamma_{z_{\rm reg}} \tag{10}$$

(i.e., the energy of gust response is bounded by $\gamma_{z_{\rm reg}}$).

Proof: From Lemma 1 we have that K_f stabilizes P if $\epsilon_a\gamma(K_f) < 1$. But $\epsilon_a\gamma(K_f) \leq \gamma_\infty'$. Hence K_f stabilizes P if $\gamma_\infty' < 1$. On the other hand, by Eq. (9) we know that

$$\|z_{\rm reg}\|_2 \leq \frac{1}{1 - \gamma_\infty'} \|z_1\|_2$$

It is also easy to see that $\|z_1\|_2 \leq \gamma_2'$. □

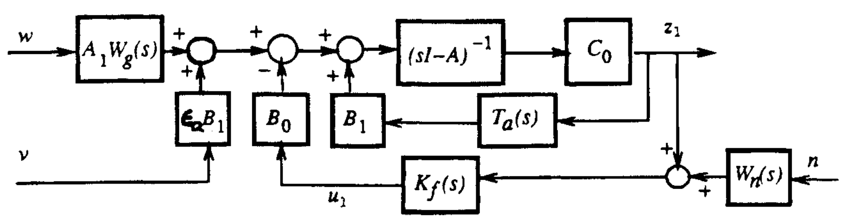

Fig. 5 Modified feedback system.

Note that as $r \to 0$ and $W_n \to 0$ we have

$$\gamma_2' \to \|z_1\|_2$$

In summary, for robust stability and gust alleviation we want to design a controller K_f so that $\gamma_\infty' < 1$, and $\gamma_{z_{\mathrm{reg}}}$ is minimized. It is obvious that $\gamma_\infty' \geq \epsilon_a \gamma_{\mathrm{opt}}$. Intuitively speaking, if we try to make γ_∞' small (i.e., close to $\epsilon_a \gamma_{\mathrm{opt}}$), then the resulting γ_2' can be very large. In this case, even though $1/(1 - \gamma_\infty')$ is small, $\gamma_{z_{\mathrm{reg}}}$ can be large. Conversely, if we try to minimize γ_2', then γ_∞' can be close to 1, and again $\gamma_{z_{\mathrm{reg}}}$ can be large. Therefore, the optimal (i.e., smallest) value of $\gamma_{z_{\mathrm{reg}}}$ corresponds to a γ_∞' value that is between $\epsilon_a \gamma_{\mathrm{opt}}$ and 1.

The problem just defined can be solved using an iterative method by computing a mixed H^2/H^∞ controller at each step. The procedure is described as follows:

1) Choose a number γ slightly larger than $\epsilon_a \gamma_{\mathrm{opt}}$.
2) Minimize γ_2' subject to the condition that $\gamma_\infty' < \gamma$.
3) Find the controller corresponding to minimal γ_2' computed in step 2.
4) Calculate the actual $\gamma_{z_{\mathrm{reg}}}$ for this controller.
5) Go to step 1, increase γ, and repeat the procedure until $\gamma = 1$.
6) Plot $\gamma_{z_{\mathrm{reg}}}$ vs γ; the minimal value of $\gamma_{z_{\mathrm{reg}}}$ is the optimal performance.

An important point to remark here is that step 2 of the previous procedure requires minimization of γ_2', under the condition $\gamma_\infty' < \gamma$. This is a true mixed H^2/H^∞ optimization problem, for which there is no simple solution. A solution to the modified version of this problem, which minimizes an upper bound of γ_2', is available.[14–17] Recall that originally we wanted to minimize $\|z_{\mathrm{reg}}\|_2$, but it was difficult to do so, and we introduced γ_2', which gives an upper bound for $\|z_{\mathrm{reg}}\|_2$. In the following numerical example, instead of minimizing γ_2', in step 2 we minimize an upper bound of γ_2' (see Refs. 14, 15, and 17 for the precise definition of this upper bound and a discussion of how conservative this bound is) and obtain a controller from this modified mixed H^2/H^∞ control problem. Then in step 4 we can compute the actual γ_2', and γ_∞' corresponding to this controller. This requires computation of the 2-norm and ∞-norm of finite dimensional transfer functions, which can be done easily.[11,12,18]

V. Numerical Example

We choose following numerical values for the parameters of the thin airfoil shown in Fig. 1 (these numerical values were taken from page 18, example 1, of Ref. 6; for more details on the physical significance of these parameters see Refs. 6 and 7):

$$
\begin{aligned}
a &= -0.400 \text{ units of } b\\
b &= 1.000 \text{ ft}\\
c &= 0.500 \text{ units of } b\\
x_\alpha &= 0.200 \text{ units of } b\\
x_\beta &= 1/80 \text{ units of } b\\
\zeta_\beta &= 0.007\\
\rho &= 0.031830989 \text{ slugs/ft}^3\\
V &= 200.0 \text{ ft/s}\\
m_s &= 1.000 \text{ slugs}\\
\omega_\alpha &= 100.0 \text{ rad/s}\\
\omega_\beta &= 90.0 \text{ rad/s}\\
\omega_h &= 60.0 \text{ rad/s}\\
r_\alpha &= 0.500 \text{ units of } b\\
r_\beta &= \sqrt{1/160} \text{ units of } b
\end{aligned}
$$

We can compute the system matrices A, B_0, B_1, and C_0 based on the equations of Sec. II. Then we need an approximation of $T(s)$; i.e., we need a rational function $T_a(s)$ and an error bound ϵ_a to obtain a finite dimensional plant model $P_f(s)$. Note that Theodorsen's function is a function of reduced frequency $s = b/V(j\omega)$. In this case, when $b = 1.000$ ft and $V = 200$ ft/s, we can define[10]

$$T_a(s) = \frac{(18.6 \times 1.000 \times s/200 + 1)(2.06 \times 1.000 \times s/200 + 1)}{(21.98 \times 1.000 \times s/200 + 1)(3.44 \times 1.000 \times s/200 + 1)}$$

It was shown that[10] for this particular choice of T_a we have $\epsilon_a = 0.012$.

Using the previous second-order stable T_a, we can now obtain our approximate plant as

$$P_f(s) = K\left[\frac{\Pi_{i=1}^{7}(s - z_i)}{\Pi_{j=1}^{8}(s - p_i)}\right]$$

where

$K = 1352.1$

$$
\begin{aligned}
z_1 &= -711.80 & p_1 &= -74.86 + 80.28i\\
z_2 &= -58.14 & p_2 &= -74.86 - 80.28i\\
z_3 &= -28.67 & p_3 &= -36.28\\
z_4 &= -9.10 & p_4 &= -22.40 + 106.81i\\
z_5 &= 28.70 & p_5 &= -22.40 - 106.81i\\
z_6 &= 0.32 + 68.08i & p_6 &= -7.96\\
z_7 &= 0.32 - 68.08i & p_7 &= 17.84 + 89.45i\\
& & p_8 &= 17.84 - 89.45i
\end{aligned}
$$

Note that the plant is nonminimum phase and unstable.

The next step is to obtain γ_{opt} and the corresponding controller K_f from the one block H^∞ optimal control problem

$$\gamma_{\mathrm{opt}} = \inf_{K_f \text{stabilizing } P_f} \|W_1(1 + P_f K_f)^{-1}\|_\infty \tag{11}$$

where $W_1 = N_1 M_f^{-1}$, which can be computed from the problem data as described in Sec. III. For our numerical example we find that

$$W_1(s) = K\left[\frac{\Pi_{i=1}^{7}(s - z_i)}{\Pi_{j=1}^{8}(s - p_i)}\right]$$

$K = 19.65$

$$
\begin{aligned}
z_1 &= -328.95 & p_1 &= -74.86 + 80.28i\\
z_2 &= -225.01 & p_2 &= -74.86 - 80.28i\\
z_3 &= 131.59 & p_3 &= -36.28\\
z_4 &= -58.14 & p_4 &= -22.40 + 106.81i\\
z_5 &= -9.10 & p_5 &= -22.40 - 106.81i\\
z_6 &= 25.33 + j34.79 & p_6 &= -7.96\\
z_7 &= 25.33 - j34.79 & p_7 &= 17.84 + 89.45i\\
& & p_8 &= 17.84 - 89.45i
\end{aligned}
$$

The problem (11) can be solved easily using one block H^∞ control techniques.[11] For this numerical example we have

$$\gamma_{\mathrm{opt}} = 2.7590$$

Note that, since P_f is strictly proper, the optimal K_f solving Eq. (11) is improper. To remedy this problem, we added two stable poles (with large magnitudes) to make the optimal controller proper and obtained an "approximately optimal" controller given as follows:

$$K_f := K\left[\frac{\Pi_{i=1}^{9}(s - z_i)}{\Pi_{j=1}^{9}(s - p_j)}\right]$$

where

$K = -311{,}500$

$$
\begin{array}{ll}
z_1 = -73.45 + 77.75i & p_1 = -4.21 \times 10^8 \\
z_2 = -73.45 - 77.75i & p_2 = -711.80 \\
z_3 = -36.05 + 124.84i & p_3 = -58.14 \\
z_4 = -36.05 - 124.84i & p_4 = -34.88 + 123.42i \\
z_5 = -36.01 & p_5 = -34.88 - 123.42 \\
z_6 = -33.03 + 123.89i & p_6 = -28.67 \\
z_7 = -33.03 - 123.89i & p_7 = -9.10 \\
z_8 = -7.83 & p_8 = 31.15 + 39.27i \\
z_9 = 96.25 & p_9 = 31.15 - 39.27i
\end{array}
$$

The controller K_f places the system poles at (note that there are several stable pole zero cancellations in the product P_fK_f, which is expected from the one block H^∞ problem)

$$
\begin{array}{ll}
p_1 = -6.85 \times 10^4 & p_7 = -43.16 - 35.41i \\
p_2 = -4.15 \times 10^4 & p_8 = -34.67 + 123.45i \\
p_3 = -414.67 & p_9 = -34.67 - 123.45i \\
p_4 = -158.64 + 32.30i & p_{10} = -21.78 + 40.80i \\
p_5 = -158.64 - 32.30i & p_{11} = -21.78 - 40.80i \\
p_6 = -43.16 + 35.41i & p_{12} = -9.27
\end{array}
$$

so it stabilizes the nominal plant P_f. Usually, whenever the original plant has right half-plane poles and zeros, the one block H^∞ controller is unstable. We see this situation in our example: an unstable controller leads to a stable closed-loop system. If the original plant was stable (airfoil is out of flutter condition) and minimum phase, then[11] we could expect a stable one block H^∞ controller.

We see from Fig. 6 that $\gamma(K_f) = 2.8982$. This performance is "fairly close" to $\gamma_{\mathrm{opt}} = 2.7590$. For this controller, K_f, the maximum approximation error (in Theodorsen's function approximation) that the system can handle without violating stability, is given by $\epsilon_a^{\max} := \gamma_{\mathrm{opt}}^{-1} = 0.345$. Recall that in our case $\epsilon_a = 0.012$ (considerably smaller than $\epsilon_a^{\max}$, so

$$\epsilon_a \gamma(K_f) \approx 0.035 \ll 1$$

and there is a large robustness margin in this design. Figure 6 also shows that the closed-loop system has high bandwith. For practical applications this situation is undesirable, because the measurement noise (which is neglected here) may be amplified.

However, in the previous discussion robust stability was our only concern. Next we will consider the gust alleviation problem, where the effect of measurement noise will also be taken into account.

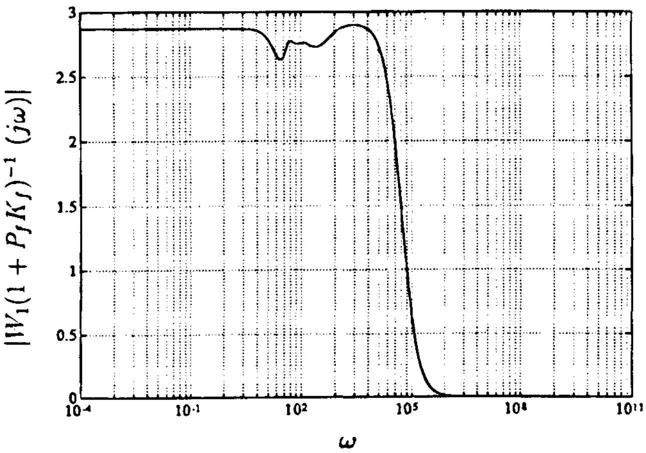

Fig. 6 $|W_1(1 + P_fK_f)^{-1}(j\omega)|$ vs ω.

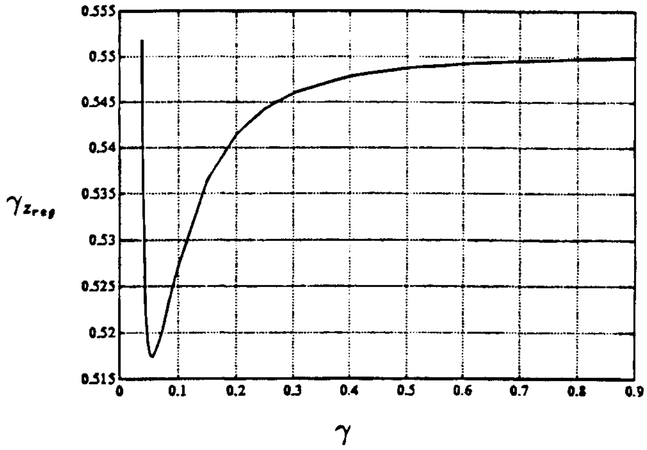

Fig. 7 $\gamma_{z_{\mathrm{reg}}}$ vs γ.

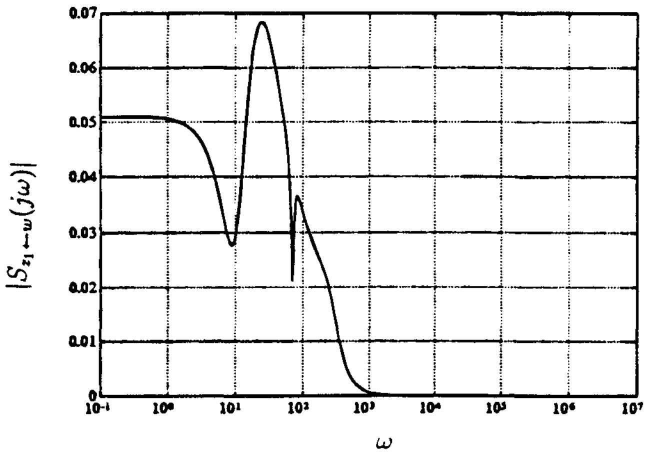

Fig. 8 Magnitude plot of $|S_{z_1 \leftarrow w}(j\omega)|$.

For our example we choose

$$\hat{W}_g(s) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \frac{20}{s + 20}, \qquad W_n(s) = \frac{0.02\, s}{s + 0.01}$$

and $r = 0.001$. Then we follow the procedure described in Sec. IV. In step 2 of our procedure, we apply the results of Refs. 14, 15, and 17 to minimize an upper bound of γ_2', subject to $\gamma_\infty' < \gamma$, for each fixed γ. A plot of $\gamma_{z_{\mathrm{reg}}}$ vs γ is given in Fig. 7 for $1 > \gamma > \epsilon_a\gamma_{\mathrm{opt}} = 0.035$. We see that the minimum occurs at $\gamma = 0.056$, and for this value of γ we find that $\gamma_\infty' = 0.0544$, $\gamma_2' = 0.4892$, and $\gamma_{z_{\mathrm{reg}}} = 0.5174$. The controller that corresponds to these numbers is given by

$$K_f(s) = K\left[\frac{\Pi_{i=1}^{9}(1 - s/z_i)}{\Pi_{j=1}^{10}(1 - s/p_j)}\right] \tag{12}$$

where

$K = -0.1249$

$$
\begin{array}{ll}
z_1 = -208.4 + 1.32j & p_1 = -1352318 \\
z_2 = -208.4 - 1.32j & p_2 = -711.8 \\
z_3 = -8.90 + 72.28j & p_3 = -146.3 \\
z_4 = -8.90 - 72.28j & p_4 = 1.77 + 68.12j \\
z_5 = -71.47 & p_5 = 1.77 - 68.12j \\
z_6 = -32.61 & p_6 = -58.14
\end{array}
$$

$$
\begin{aligned}
z_7 &= -17.62 & p_7 &= -28.67 \\
z_8 &= -7.82 & p_8 &= -20.00 \\
z_9 &= -0.01 & p_9 &= -9.10 \\
& & p_{10} &= -0.0108
\end{aligned}
$$

This controller guarantees that 1) the closed-loop system, with finite dimensional controller K_f and the infinite dimensional plant P, is stable because 2) $\epsilon_a \gamma(K_f) = \|S_{z1\leftarrow v}\|_\infty < \gamma'_\infty = 0.0544 \ll 1$, and 3) $\|S_{z_{reg}\leftarrow w}\|_2 < \gamma_{z_{reg}} = 0.5174$.

A plot of $|S_{z1\leftarrow w}(j\omega)|$ vs ω is shown in Fig. 8. This figure shows that the closed-loop system passes only the low-frequency (less than 10^3 rad/s) content of the disturbance w with an amplification less than 0.07 (i.e., the attenuation level is greater than $1/0.07 \approx 14 \approx 23$ dB).

VI. Concluding Remarks

In this paper we have considered a thin airfoil model, which is infinite dimensional. We have shown that a finite dimensional controller stabilizing this system can be obtained by solving a one block H^∞ control problem. We have determined a sufficient condition for such a controller to stabilize the infinite dimensional plant. This condition depends on the finite dimensional plant and on the error in the approximation of Theodorsen's function. We have shown a numerical example illustrating this approach.

The gust alleviation problem is formulated in the framework of the mixed H^2/H^∞ control problem, where the plant to be controlled is finite dimensional. In this paper we developed a procedure to solve this problem and demonstrated our approach with a numerical example.

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